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On Extremal Multiflows

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Given an Eulerian multigraph, a subset T of its vertices, and a collection $\mathscr{H}$ of subsets of T, we ask how few edge-disjoint paths can contain maximum $(A, T\backslash A)$-flows, for all $A \in \mathscr{H}$ at once. We answer the question for a certain class of hypergraphs $\mathscr{H}$ by presenting a strongly polynomial construction of a minimum set of such paths and a min-max formula for its cardinality. The method consists in reducing the problem to maximizing a b-matching in some graph. The result provides a solution to one interesting class of path packing problems. © 2000 Academic Press

1. INTRODUCTION

Throughout the paper, we mean by *graph* an undirected multigraph and by *network* a pair (G, T) consisting of a graph G and a subset T of its vertices, $|T| \geqslant 2$. The vertices from T are called *terminals*; the other vertices are *inner*. The question posed in the paper sounds as follows. Given a collection $\mathscr{H}$ of proper subsets of T, how small can a set of edge-disjoint paths in G which contains maximum $(A, T\backslash A)$-flows be, for all $A \in \mathscr{H}$ at

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once? We answer this question under certain natural conditions on G and $\mathcal{H}$ guaranteeing the existence of such a set of paths.

Except for Section 3, the graphs we deal with are Eulerian.

Some notations and basic notions are needed to pose the problem in exact terms and to state the result. The vertex set of G is denoted by V, and we denote by $\bar{A}$ the complement $V \backslash A$ of a subset $A \subseteq V$ and by A^c the complement $T \backslash A$ of a subset $A \subseteq T$. We denote by $d(v)$ the degree of a vertex v. For a subset $X \subseteq V$, we denote by $d(X)$ the cardinality of the *cut* generated by X, that is, the number of edges with exactly one end in X; the term *cut* will often mean the set X too. When the graph G is to be indicated, we write d_G instead of d.

If **u** is a function (or vector) defined on some set, and X is a finite subset, we write $u[X]$ for the sum $\sum_{x \in X} u(x)$. According to this rule, we have for $X \subseteq V$

$$d[X] := \sum_{v \in X} d(v) = d(X) + 2 \times \text{the number of edges with both ends in } X. \tag{1}$$

A *T-path* is a path in G whose ends are distinct terminals; by *multiflow* (or *T-flow*) we mean in this paper a collection of edge-disjoint T-paths (i.e., what is usually called integer multiflow). For $A \subseteq T$, an (A, A^c)*-flow* is a multiflow whose paths have one end in A and the other in A^c. A multiflow *locks* A if it contains a maximum (A, A^c)-flow and *locks* a hypergraph $\mathcal{H}$ on T if it locks each $A \in \mathcal{H}$. Our question may be now formulated as follows.

Problem 0 (minimum locking). Given a network (G, T) and a hypergraph $\mathcal{H}$ on T, what is the least size of a multiflow locking $\mathcal{H}$?

This minimum will be denoted by σ.

Two reasons for a hypergraph to be unlockable in a given network are illustrated by the following examples, in each of which the network is a star formed by k terminals t_i, $i = 1, ..., k$, linked to the single inner vertex, each by one edge (Fig. 1).

EXAMPLE 1. k odd (say, $k = 3$). The singletons $\{t_i\}$, $i = 1, ..., k$, cannot be locked at once. The reason for that is in the network rather than in the hypergraph $\mathcal{H}$ which is as simple as it ever could be. Indeed, by doubling the edges we obtain a graph in which the k singletons are lockable simultaneously.

EXAMPLE 2. $k = 4$. The hypergraph consisting of the pairs $\{t_4, t_i\}$, $i = 1, 2, 3$, cannot be locked, even after multiplying the edges, the reason for

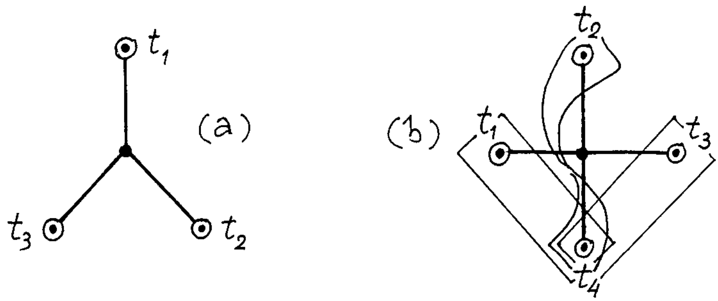

FIG. 1. (a) G has inner vertex of odd degree; (b) $\mathscr{H}$ contains a 3-cross.

which may be attributed to the presence of a 3-cross in the hypergraph (see below).

A graph is called *Eulerian* if it has only even degrees, and *inner Eulerian* if the degrees of all inner vertices are even.

Two sets, A and B, form a *semicross* (or are *semicrossing*) if $A \backslash B$, $B \backslash A$, and $A \cap B$ are nonempty. Semicrossing subsets $A, B \subseteq T$ are *crossing* if $(A \cup B)^c$ is also nonempty (as in Example 2). Three subsets of T form a 3-*cross* (a 3-semicross) if any two of them are crossing (respectively, semicrossing).

Let us now call a hypergraph $\mathscr{H}$ *lockable* if a multiflow locking $\mathscr{H}$ exists in every inner Eulerian network (G, T) (cf. [10]).

THEOREM 1.1 (Karzanov and Lomonosov [10]). *A hypergraph is lockable iff it contains no* 3-*cross.*

We call a hypergraph $\mathscr{H}$ on T *discrete* if for any $A \in \mathscr{H}$ and $t \in A^c$ there is a set $B \in \mathscr{H}$ such that $t \in B \subseteq A^c$.

$\mathscr{H}$ is called k-*regular* if each $t \in T$ belongs to exactly k sets of $\mathscr{H}$. It is an easy exercise to prove that a regular 3-semicross free hypergraph is discrete. Important for us in this work are 2-regular hypergraphs, that is, the hypergraph duals of graphs. For them the property of being discrete may be stated in two other equivalent forms.

CLAIM 1.2. *For a* 2-*regular hypergraph* $\mathscr{H}$, *the following three statements are equivalent*;

(1.2.1) $\mathscr{H}$ *is discrete*;

(1.2.2) $\mathscr{H}$ *has no* 3-*semicross*;

(1.2.3) *the graph* $\mathscr{H}^*$ *has no triangle.*

Here $\mathscr{H}^*$ means the hypergraph dual to $\mathscr{H}$, and the term "graph" includes multigraphs.

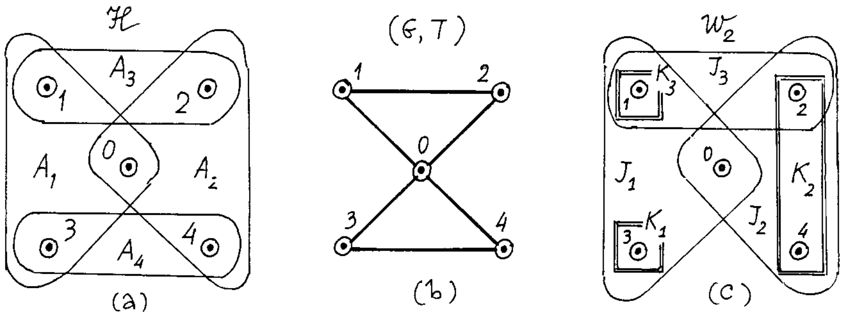

FIG. 2. 2-regular 3-cross free nondiscrete hypergraph.

It may be worth emphasizing that a 3-semicross free hypergraph is also 3-cross free and hence lockable. The following example illustrates the above notions.

EXAMPLE 3. (See Fig. 2.) Consider the set $T = \{0, 1, 2, 3, 4\}$ of terminals, the 2-regular hypergraph $\mathcal{H}$ on T consisting of the four sets $A_1 = \{0, 1, 3\}$, $A_2 = \{0, 2, 4\}$, $A_3 = \{1, 2\}$, and $A_4 = \{3, 4\}$ (Fig. 2a), and the network with the vertex-set T and the six edges $(1, 2)$, $(3, 4)$, and $(0, i)$, $i = 1, 2, 3, 4$ (Fig. 2b). $\mathcal{H}$ is not discrete (see, e.g., the set A_1 and terminal 2). By (2), it contains semicrosses; one of them is formed by the sets A_1, A_2, and A_3. Since, however, $\mathcal{H}$ has no 3-cross, it is lockable, by Theorem 1.1. In the given network, $\mathcal{H}$ is locked by the multiflow formed by the paths $(1, 2)$, $(3, 4)$, $(1, 0, 4)$, and $(2, 0, 3)$.

In this paper Problem 0 is solved for inner Eulerian networks and 2-regular discrete hypergraphs $\mathcal{H}$. To simplify the presentation, the main theorem is stated for entirely Eulerian graphs. This result is proved in Section 2, by reduction to maximization of a balanced flow [7], and extended to the inner Eulerian networks in Section 3. In Section 4 we show that the solved 2-regular case of Problem 0 majorates one class of paths packing problems. We proceed now to formulate the main result of the paper.

DEFINITION. Given a triple $(G, T, \mathcal{H})$, a family $\mathcal{W}$ of subsets of V is called *kernel* 2-*cover* if

- it consists of as many as $2|\mathcal{H}|$ sets, $K_A, J_A \subseteq V$, $A \in \mathcal{H}$, not necessarily distinct (more precisely, $\mathcal{W}$ should be considered as a function $\mathcal{H} \times \{K, J\} \to 2^V$ where K and J are some symbols);
- for any $A \in \mathcal{H}$ there hold the relations $\emptyset \subseteq K_A \subseteq J_A$ and $J_A \cap T \subseteq A$;
- each terminal belongs to at least two members of $\mathcal{W}$.

The third condition is trivially satisfied when a terminal belongs to some K_A; otherwise, it requires that distinct members A and B of $\mathscr{H}$ should exist such that $t \in (J_A \backslash K_A) \cap (J_B \backslash K_B)$ (these A and B may, however, consist of the same terminals). The *capacity* of a kernel 2-cover $\mathscr{W}$ is defined by

$$\|\mathscr{W}\| := \frac{1}{4} \sum_{X \in \mathscr{W}} d(X) - \frac{\omega}{2}, \tag{2}$$

where ω is the number of odd $\mathscr{W}$-components (see below).

Given a triple $(G, T, \mathscr{H})$, we define $\lambda(A) := \min\{d(X) : X \subset V,\ X \cap T = A\}$, for any proper subset A of T, and put $\Lambda := \frac{1}{2} \sum_{A \in \mathscr{H}} \lambda(A)$.

Main theorem. *Let (G, T) be a network and $\mathscr{H}$ be a hypergraph on T. If G is Eulerian, and $\mathscr{H}$ is 2-regular and discrete then*

$$\sigma = \Lambda - \min \|\mathscr{W}\|, \tag{3}$$

the minimum over the kernel 2-covers $\mathscr{W}$.

It remains to define the $\mathscr{W}$-components and their parity. Given a kernel 2-cover $\mathscr{W}$, for each $A \in \mathscr{H}$ consider the induced subgraph $G(J_A) - K_A$, and let $\mathscr{Y}_A = \{Y_A^1, Y_A^2, ...\}$ be the collection of its connectivity components. Let us merge the collections $\mathscr{Y}_A$, preserving repetitions if any, so that the resulting family, denoted by $\mathscr{Y}$, consists of all the components Y_A^i, $A \in \mathscr{H}$, $i = 1, 2, ...$. For $(i, A) \neq (j, B)$, we define the members Y_A^i and Y_B^j of $\mathscr{Y}$ to be *adjacent* if $Y_A^i \cap Y_B^j \cap T \neq \varnothing$ (which obviously implies $A \neq B$ and $A \cap B \neq \varnothing$). A component of $\mathscr{Y}$ under this adjacency relation will be called a *$\mathscr{W}$-component*.

To define the $\mathscr{W}$-component parity, let us write the sum $\sum_{X \in \mathscr{W}} d(X)$ in the form

$$\sum_A (d(K_A) + d(J_A)) = 2 \sum_A d(K_A) + \sum_A (d(J_A) - d(K_A)).$$

The difference in the latter term may be expressed through the components of $G(J_A) - K_A$, in the form $\sum_{Y \in \mathscr{Y}_A} \delta(Y)$ where $\delta(Y) := (d(Y \cup K_A) - d(K_A))$ whence

$$\sum_A (d(J_A) - d(K_A)) = \sum_{Y \in \mathscr{Y}} \delta(Y).$$

Let us now introduce the quantity

$$p(\mathscr{C}) := \tfrac{1}{2} \sum_{Y \in \mathscr{C}} \delta(Y), \tag{4}$$

for any $\mathscr{W}$-component $\mathscr{C}$. Since G is Eulerian, $p(\mathscr{C})$ is always an integer. In these terms we may write

$$\frac{1}{4}\sum_{X\in\mathscr{W}} d(X)=\frac{1}{2}\sum_{A\in\mathscr{H}} d(K_A)+\sum_{\mathscr{C}}\frac{1}{2}\,p(\mathscr{C}).$$

We call a $\mathscr{W}$-component $\mathscr{C}$ *odd* iff $p(\mathscr{C})$ is an odd number. Then the kernel 2-cover capacity given by (2) coincides with the integer expression

$$\|\mathscr{W}\|=\frac{1}{2}\sum_{A\in\mathscr{H}} d(K_A)+\sum_{\mathscr{C}}\left\lfloor\frac{1}{2}\,p(\mathscr{C})\right\rfloor, \tag{5}$$

which has initially motivated the notions of a $\mathscr{W}$-component and its parity.

The condition imposed on the hypergraph $\mathscr{H}$ by the main theorem is stronger than that of Theorem 1.1. This condition, meaning, by Claim 1.2, the absence of 3-semicrosses, cannot be relaxed to only having no 3-cross; to show this, we return to the above example.

Example 3 (continued). Let us first illustrate the notion of kernel 2-cover by two families of sets. We write K_i instead of K_{A_i}, and similarly for J.

$\mathscr{W}_1$: $K_i=\varnothing$ and $J_i=A_i$, $i=1, ..., 4$. We have $\frac{1}{4}\sum_{X\in\mathscr{W}_1} d(X)=3$; the single $\mathscr{W}_1$-component consists of the sets J_i, $i=1, ..., 4$, and is even. Thus, $\|\mathscr{W}_1\|=3$.

$\mathscr{W}_2$: $K_1=\{3\}$, $K_2=\{2, 4\}$, $K_3=\{1\}$, and $J_i=A_i$, $i=1, 2, 3$; $K_4=J_4=\varnothing$. We have $\frac{1}{4}\sum_{X\in\mathscr{W}_2} d(X)=\frac{9}{2}$, and there are two $\mathscr{W}_2$-components, inevitably of different parity, so that $\|\mathscr{W}_2\|=4$. The odd $\mathscr{W}_2$-component consists of the sets $J_{A_1}\backslash K_{A_1}=\{0, 1\}$ and $J_{A_2}\backslash K_{A_2}=\{0\}$; the even one is formed by the single set $J_{A_3}\backslash K_{A_3}=\{1\}$.

By enumerating the possible cases, one can verify that 3 is indeed the minimum value of kernel 2-cover capacity in our case. Further, we obviously have $\Lambda=6$ whence $\Lambda-\min\|\mathscr{W}\|=3$. On the other hand, the four paths (1, 2), (3, 4), (1, 0, 4), and (2, 0, 3) are easily seen to form a minimum multiflow locking $\mathscr{H}$, so that $\sigma=4$. Thus, $\sigma>\Lambda-\min\|\mathscr{W}\|$; we see that the main theorem does not hold in our example.

2. PROOF OF MAIN THEOREM

In Subsections 2.1–2.2, the graph G is Eulerian, and the hypergraph $\mathscr{H}$ is discrete and 3-cross free, with arbitrary degrees. Under these assumptions we prove Theorem 2.3 which implies that the requirement of Problem 0, that the members of $\mathscr{H}$ be locked by the same multiflow, is decomposable into independent constraints related to the members of $\mathscr{H}$.

Together with Theorem 1.1, this result enables reducing Problem 0 to maximizing a balanced flow; this is done in Subsections 2.3–2.4. In the present paper this reduction is confined to 2-regular hypergraphs only.

2.1. *Cuts and Flows*

Here we list some relevant properties of network cuts. The set-function $d(X)$, $X \subseteq V$ (see Introduction) satisfies the inequality

$$d(X) + d(Y) \geqslant d(X \setminus Y) + d(Y \setminus X), \qquad \text{for every} \quad X, Y \subseteq V \tag{6}$$

which, due to the symmetry $d(X) = d(\bar{X})$, is equivalent to the submodularity condition (see, e.g., [13]).

Earlier, we have denoted by $\lambda(A)$ the maximum size of an (A, A^c)-flow, $\emptyset \subset A \subset T$. Given a *demand* vector $\mathbf{q} \in \mathbb{Z}_+^T$, two versions of the degree-constrained maximum flow problem can be posed, giving rise to two more set-functions on V. Let $d_{\mathscr{F}}(t)$ denote the number of paths of a multiflow $\mathscr{F}$ having an end in a terminal t, and let $\mathbf{d}_{\mathscr{F}} := (d_{\mathscr{F}}(t) : t \in T)$ be called the *degree vector* of $\mathscr{F}$.

Choose $A \subset T$. The maximum size of an (A, A^c)-flow $\mathscr{F}$, subject to the degree constraints $d_{\mathscr{F}}(t) \leqslant q(t)$, $t \in T$, equals the minimum cardinality of an $(A', (A^c)')$-cut in the extended network (G', T'), where T' is a disjoint copy of T, $A' \subseteq T'$ is the copy of A, and G' is constructed by linking each $t \in T$ to its copy $t' \in T'$, by as many as $q(t)$ parallel edges. An $(A', (A^c)')$-cut in G' is generated in the usual way by a set of vertices of the form $A' \cup X$ where X is an arbitrary subset of V. Since such a cut is completely determined by a subset $X \subseteq V$, we refer to X as an $(A, \mathbf{q})$-*cut* and denote its capacity by

$$d(X \mid \mathbf{q}) := d_{G'}(A' \cup X) = d(X) + q[A \setminus X] + q[A^c \cap X]. \tag{7}$$

The minimum capacity of an $(A, \mathbf{q})$-cut will be denoted by $\lambda(A, \mathbf{q})$.

Suppose now that an (A, A^c)-flow is maximized under the partial constraints

$$d_{\mathscr{F}}(t) \leqslant q(t), \qquad t \in A. \tag{8}$$

Again, we implement the degree constraints by appending a disjoint copy A' of the set A and connecting each $t \in A$ to its copy $t' \in A'$ by as many as $q(t)$ parallel edges. An (A', A^c)-cut in the extended graph has the form $A' \cup X$ where X is an arbitrary subset of $V \setminus A^c$; we call X an $(A, \mathbf{q}_A)$-cut and denote its capacity by $d(X \mid \mathbf{q}_A)$. By the above definition,

$$d(X \mid \mathbf{q}_A) = d(X) + q[A \setminus X]. \tag{9}$$

The minimum capacity of an $(A, \mathbf{q}_A)$-cut will be denoted by $\lambda(A, \mathbf{q}_A)$. The latter notations point at the possibility of considering (8) as the overall constraints $\mathbf{d}_{\mathscr{F}} \leqslant \mathbf{q}_A$, with the demand vector $\mathbf{q}_A \in \mathbb{Z}_+^T$ coinciding with $\mathbf{q}$ in A and equal to infinity (i.e., a sufficiently large number) in A^c.

The relation $\mathbf{q} \leqslant \mathbf{q}_A \leqslant \infty$ implies

$$\lambda(A, \mathbf{q}) \leqslant \lambda(A, \mathbf{q}_A) \leqslant \lambda(A). \tag{10}$$

The following property of cuts intersection takes place in arbitrary graphs, not necessarily Eulerian.

LEMMA 2.1. *Let (G, T) be a network and A, B be disjoint proper subsets of T. Given a demand vector $\mathbf{q}$, let X be a minimum $(A, \mathbf{q})$-cut and Y be a minimum $(B, \mathbf{q}_B)$-cut. Then $X \backslash Y$ is a minimum $(A, \mathbf{q})$-cut, and $Y \backslash X$ is a minimum $(B, \mathbf{q}_B)$-cut.*

Proof. By the definition of $(B, \mathbf{q}_B)$-cut, we have $Y \cap T \subseteq B$, whence $A \backslash (X \backslash Y) = A \backslash X$ and $A^c \cap (X \backslash Y) = (A^c \cap X) \backslash (A^c \cap X \cap Y) = (A^c \cap X) \backslash (B \cap X \cap Y)$. Therefore, the capacity of the $(A, \mathbf{q})$-cut generated by the set $X \backslash Y$ equals, by (7),

$$d(X \backslash Y \mid \mathbf{q}) = d(X \backslash Y) + q[A \backslash X] + q[A^c \cap X] - q[B \cap X \cap Y],$$

and the inequality $d(X \backslash Y \mid \mathbf{q}) \geqslant d(X \mid \mathbf{q})$ implies

$$d(X \backslash Y) \geqslant d(X) + q[B \cap X \cap Y]. \tag{11}$$

On the other hand, the capacity of the $(B, \mathbf{q}_B)$-cut generated by $Y \backslash X$ equals, by (9),

$$\begin{aligned} d(Y \backslash X \mid \mathbf{q}_B) &= d(Y \backslash X) + q[B \backslash (Y \backslash X)] \\ &= d(Y \backslash X) + q[B \backslash Y] + q[B \cap X \cap Y], \end{aligned}$$

and the inequality $d(Y \backslash X \mid \mathbf{q}_B) \geqslant d(Y \mid \mathbf{q}_B)$ implies

$$d(Y \backslash X) \geqslant d(Y) - q[B \cap X \cap Y]. \tag{12}$$

Combining (11) and (12) we obtain the inequality

$$d(X \backslash Y) + d(Y \backslash X) \geqslant d(X) + d(Y),$$

which, together with the submodularity relation (6), implies that the inequalities (11) and (12) hold with equalities. This, in turn, means that $d(X \backslash Y \mid \mathbf{q}) = d(X \mid \mathbf{q})$ and $d(Y \backslash X \mid \mathbf{q}_B) = d(Y \mid \mathbf{q}_B)$. ∎

Following [2, 3], we say that a multiflow $\mathscr{F}$ **q**-*locks* a subset $A \subset T$ if it contains a maximum (A, A^c)-flow, subject to the demand constraints $d_{\mathscr{F}}(t) \leqslant q(t)$, for all $t \in T$, and **q**-*locks* a hypergraph $\mathscr{H}$ on T if it **q**-locks each $A \in \mathscr{H}$.

A multiflow with all the degrees even will be called *Eulerian*. In this section we confine ourselves to Eulerian multiflows only, due to the following property established in the Appendix, Claim 5.1, in a slightly more general form.

CLAIM 2.2. *Let G be Eulerian and $\mathscr{H}$ be a 3-cross free hypergraph on T. Then for every multiflow locking $\mathscr{H}$ there is an Eulerian multiflow of the same size, which also locks $\mathscr{H}$.*

2.2. *Decomposition of the Locking Constraint*

Our aim here is to simplify the constraint of Problem 0, in the case when G is Eulerian and $\mathscr{H}$ is discrete and 3-cross free (but not necessarily regular). We denote by $\mathscr{H}_t$ the collection of sets $A \in \mathscr{H}$ containing a terminal t.

Recall that we deal with Eulerian multiflows only, due to Claim 2.2.

It may be noticed that Problem 0 actually involves only multiflow degrees. To express this explicitly, we call a vector $\mathbf{x} \in \mathbb{Z}_+^T$ *feasible* (with respect to given G, T, $\mathscr{H}$) if $2\mathbf{x}$ majorates the degree vector of an Eulerian multiflow locking $\mathscr{H}$. Let the set of feasible vectors be denoted by $\mathbf{F}$. Then Problem 0 is equivalent to minimizing the total of a vector in $\mathbf{F}$, that is

$$\sigma = \min\{\mathbb{1} \cdot \mathbf{x} : \mathbf{x} \in \mathbf{F}\}. \tag{13}$$

On the other hand, for any proper $A \subset T$, let a vector $\mathbf{z} \in \mathbb{Z}_+^A$ be called *A-base* if there is a maximum (A, A^c)-flow $\mathscr{F}$ such that $d_{\mathscr{F}}(t) = 2z(t)$ for $t \in A$. Let us, further, say that a vector $\mathbf{y} \in \mathbb{Z}_+^T$ *spans A* if it majorates some A-base and spans $\mathscr{H}$ if it spans each member of $\mathscr{H}$. In other words, $\mathbf{y}$ spans $\mathscr{H}$ iff $\lambda(A, 2\mathbf{y}_A) = \lambda(A)$ for each $A \in \mathscr{H}$ (cf. (10)). Let $\mathbf{H}$ denote the set of vectors spanning $\mathscr{H}$; clearly, $\mathbf{F} \subseteq \mathbf{H}$.

THEOREM 2.3. *If G is Eulerian and $\mathscr{H}$ is 3-cross free and discrete then* $\mathbf{F} = \mathbf{H}$.

This assertion is the only point in the proof of the main theorem where $\mathscr{H}$ is needed to be discrete. It should then be concluded from Example 3 (see Introduction) that $\mathscr{H}$ being discrete is essential for Theorem 2.3 too. To see this directly, return to Example 3 and consider the vector $\mathbf{y}$ with $y(i) = 1$ if $i = 0, 2, 3$ and 0 otherwise. It spans $\mathscr{H}$, but there is no multiflow locking $\mathscr{H}$ whose degrees are majorated by $2\mathbf{y}$. Indeed, for the set $A = \{0, 1, 3\}$ we have $\lambda(A, 2\mathbf{y}) = 2 < 4 = \lambda(A)$.

Proof of Theorem 2.3. The assertion will be proved if for every vector $\mathbf{y}$ spanning $\mathscr{H}$ we find a feasible vector $\mathbf{x}$ saisfying $\mathbf{x} \leqslant \mathbf{y}$. In other words, we are to show that an Eulerian multiflow $2\mathbf{y}$-locking $\mathscr{H}$ locks it also in the usual sense. This, in turn, is the same as to prove the equality $\lambda(A, 2\mathbf{y}) = \lambda(A)$ for all $A \in \mathscr{H}$. Suppose therefore that $\lambda(A, 2\mathbf{y}) < \lambda(A)$ for some $A \in \mathscr{H}$. Since $\mathbf{y}$ spans A, this means that $\lambda(A, 2\mathbf{y}) < \lambda(A, 2\mathbf{y}_A)$, by (10). From the definitions of the cuts involved (preceeding the formulas 7 and 9) we conclude that the latter inequality can hold only if every minimum $(A, 2\mathbf{y})$-cut meets A^c.

Choose a minimum $(A, 2\mathbf{y})$-cut X whose intersection with A^c is inclusion-minimal, and suppose there is $t \in X \cap A^c$. Since $\mathscr{H}$ is discrete, there exists $B \in \mathscr{H}$ such that $t \in B \subseteq A^c$. Let Y be a minimum B-cut (see Fig. 3). Since $\mathbf{y}$ spans B, Y is also a minimum $(B, 2\mathbf{y}_B)$-cut. Then, by Lemma 2.1, $X \setminus Y$ is a minimum $(A, 2\mathbf{y})$-cut.

By the definition of B-cut, we have $t \in B \subseteq Y$ whence $A^c \cap (X \setminus Y) = (A^c \cap X) \setminus B \subset A^c \cap X$ (properly), contradicting the choice of X. ∎

Theorem 2.3 suggests replacing the constraint $\mathbf{x} \in \mathbf{F}$ in the formulation (13) of Problem 0 with the condition $\mathbf{x} \in \mathbf{H}$ which simply says that $\mathbf{x}$ spans the sets $A \in \mathscr{H}$, not requiring the corresponding maximum (A, A^c)-flows to form a multiflow. To explicitly carry out this decomposition, recall that $\mathbf{x} \in \mathbf{H}$ means that in each set $A \in \mathscr{H}$ the vector $\mathbf{x}$ majorates some A-base, say $\mathbf{z}_A$. A collection of bases $\mathbf{z}_A$, $A \in \mathscr{H}$, being chosen, no other majorating vector in $\mathbf{H}$ should be considered but only that given by $x(t) := \max\{z_A(t) : A \in \mathscr{H}_t\}$. In other words, σ coincides with the minimum attained in the following

Problem 1. Given an Eulerian network (G, T), and a discrete 3-cross free hypergraph $\mathscr{H}$ on T, find a collection of bases $(\mathbf{z}_A : A \in \mathscr{H})$ minimizing the sum

$$\sum_{t \in T} \max\{z_A(t) : A \in \mathscr{H}_t\}. \tag{14}$$

2.3. *Reduction to Balanced Flows*

Here $\mathscr{H}$ is assumed to be 2-regular and discrete, as in the main theorem, and the two members of $\mathscr{H}_t$ will be denoted by A'_t, A''_t. The objective (14) of Problem 1 can then be written in the form

$$\Lambda - \sum_{t \in T} \min\{z_{A'_t}(t), z_{A''_t}(t)\}, \tag{15}$$

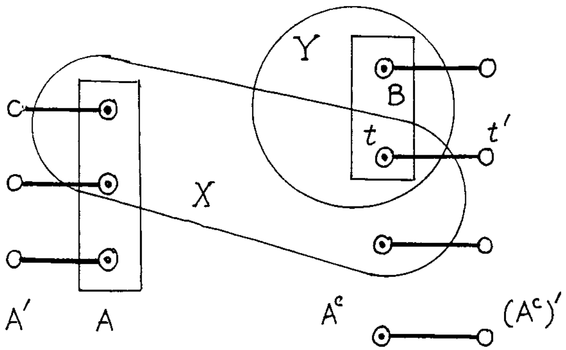

FIG. 3. To the proof of Theorem 2.3.

by using the identity $\max\{a, b\} + \min\{a, b\} = a + b$ and the relation $z_A[A] = \frac{1}{2}\lambda(A)$. Thus, for 2-regular discrete hypergraphs, $\Lambda - \sigma$ equals the maximum attained in the following

Problem 1′. Given an Eulerian network (G, T), and a 2-regular discrete hypergraph $\mathscr{H}$, maximize the sum

$$\sum_{t \in T} \min\{z_{A'_t}(t), z_{A''_t}(t)\} \tag{16}$$

over A-bases $\mathbf{z}_A$, $A \in \mathscr{H}$.

In the earlier version [6] of this work this problem is treated in terms of polymatroid matchings. After the polymatroid at hand was found to be dually representable by **b**-matchings (see [7, Theorem 2.3]), we can now translate Problem 1′ into maximizing a balanced flow (see Problem 2 below) in some other Eulerian graph $\tilde{G}$. The latter problem is solved in [7] by reducing it to maximizing a **b**-matching.

The notion of balanced flow arises when we are given a graph, $\tilde{G}$, one of whose vertices, s, is specified as the *sink*, and a partial pairing of the other vertices is fixed. So, let the vertex-set of $\tilde{G}$ be $\tilde{V} \cup \{s\}$, U be a set of disjoint pairs of vertices of $\tilde{V}$, and $\tilde{T}$ denote the union of these pairs (the set of *sources*). Then a $(\tilde{T}, s)$-flow $\mathscr{F}$ is called *balanced* if the equality $d_{\mathscr{F}}(v') = d_{\mathscr{F}}(v'')$ holds for each pair $(v', v'') \in U$.

Problem 2 (maximum Eulerian balanced flow [7]). Given an Eulerian graph with the vertex-set $\tilde{V} \cup \{s\}$ and a partial pairing U of $\tilde{V}$, what is the maximal size of a balanced Eulerian $(\tilde{T}, s)$-flow, where $\tilde{T}$ is the union of the pairs from U?

The familiar fact that any network flow is degree-majorated by some maximum flow takes place also when the network and flows are assumed to be Eulerian. So, in Problem 2 we may deal with an Eulerian maximum flow, searching to maximize its balanced subset. If we denote by $2\mathbf{z}$ the source degree vector of an Eulerian $(\tilde{T}, s)$-flow then the size of its balanced subset equals four times the sum

$$\beta(\mathbf{z}) := \sum_{(t', t'') \in U} \min\{z(t'), z(t'')\}. \tag{17}$$

Thus, Problem 2 is equivalent to maximizing $\beta(\mathbf{z})$ over the vectors $\mathbf{z} \in \mathbb{Z}_+^{\tilde{T}}$ satisfying the Gale condition

$$2 \cdot z[A] \leqslant d(X), \qquad \text{for every} \quad A \subseteq \tilde{T} \quad \text{and} \quad A \subseteq X \subseteq \tilde{V}, \tag{18}$$

and having the maximal value of $z[\tilde{T}]$.

To reduce Problem 1′ to Problem 2, we construct an input $(\tilde{G}, U, s)$ for the latter one, which will be referred to as a *spread* of $(G, T, \mathscr{H})$. Let us choose for each $A \in \mathscr{H}$ a subset $V_A \subset V$ satisfying $V_A \cap T = A$ and containing a minimum A-cut and consider the graph G_A obtained from G by shrinking the complementary subset $\overline{V_A}$ into a single vertex, s_A. Regarding the graphs G_A as disjoint, we define $\tilde{G}$ as their union in which the vertices s_A, $A \in \mathscr{H}$, are identified into a *sink* s. Each initial terminal $t \in T$ is represented in $\tilde{G}$ by two copies, t' and t'', belonging to $G_{A_t'}$ and $G_{A_t''}$, respectively. Now, $\tilde{V}$ is the set of vertices of $\tilde{G}$ distinct from s (i.e., the union of disjoint copies of V_A, $A \in \mathscr{H}$), U is the set of pairs (t', t''), $t \in T$, and $\tilde{T}$ is their union.

Note that the graph $\tilde{G}$ is, in general, not unique but depends on the choice of the subsets V_A. The two extremal cases are of special interest: the inclusion-minimal sets V_A are best for the algorithm (see below in this section) while for the proof of the main theorem (in subsection 2.4) we need them to be maximal.

The following assertion is straightforward.

THEOREM 2.4. *Let G be Eulerian, $\mathscr{H}$ be 2-regular and discrete, and let $(\tilde{G}, U, s)$ be a spread of $(G, T, \mathscr{H})$. Then*

(2.4.1) *A collection $(\mathbf{z}_A : A \in \mathscr{H})$ of A-bases forms a solution of Problem 1′ iff there exists a maximum balanced Eulerian $(\tilde{T}, s)$-flow $\mathscr{F}$ in $\tilde{G}$ such that $2z_A(t) \geqslant d_{\mathscr{F}}(t)$, for every $t \in A \in \mathscr{H}$, and*

(2.4.2) *$\Lambda - \sigma = \frac{1}{4} \max |\mathscr{F}|$, the maximum over the balanced Eulerian $(\tilde{T}, s)$-flows in $\tilde{G}$.*

Proof. We need only to summarize the above observations. If $(\mathbf{z}_A : A \in \mathscr{H})$ are A-bases then for each A there exists a maximum (A, A^c)-flow $\mathscr{F}_A$ in the graph G, having $d_{\mathscr{F}_A}(t) = 2z_A(t)$, $t \in A$. When restricted to the respective subgraphs G_A, these flows form an Eulerian $(\tilde{T}, s)$-flow in $\tilde{G}$ which we denote by $\mathscr{F}_0$. Let $\mathscr{F}$ denote the maximal balanced subset of $\mathscr{F}_0$. Then

$$|\mathscr{F}| = 2 \sum_{(t', t'') \in U} \min\{d_{\mathscr{F}_0}(t'), d_{\mathscr{F}_0}(t'')\} = 4 \sum_{t \in T} \min\{z_{A'_t}(t), z_{A''_t}(t)\}.$$

It follows that $\max |\mathscr{F}| \geqslant 4 \max \sum_t \min\{z_{A'_t}(t), z_{A''_t}(t)\} = 4(\Lambda - \sigma)$, the maxima taken over all Eulerian $(\tilde{T}, s)$-flows in $\tilde{G}$ (on the left) and over all collections of A-bases (on the right).

Conversely, for every balanced Eulerian $(\tilde{T}, s)$-flow $\mathscr{F}$ in $\tilde{G}$ and each set $A \in \mathscr{H}$ there is an A-base $\mathbf{z}_A$ majorating $\frac{1}{2}\mathbf{d}_{\mathscr{F}}$ in A, so that $|\mathscr{F}| = 2 \sum_{t \in T} d_{\mathscr{F}}(t) \leqslant 4 \sum_{t \in T} \min\{z_{A'_t}(t), z_{A''_t}(t)\}$, whence $\max |\mathscr{F}| \leqslant 4(\Lambda - \sigma)$. ∎

The theorem suggests the following way of solving Problem 0.

Step 1. Construct a spread $(\tilde{G}, U, s)$.

Step 2. Find the half-degree vector of a maximum balanced Eulerian flow in $\tilde{G}$, and for each $A \in \mathscr{H}$ construct an A-base, z_A, majorating this vector in the set A. Form the demand vector $\mathbf{q}$ by assigning

$$q(t) := 2 \max\{z_{A'_t}(t), z_{A''_t}(t)\}, t \in T.$$

(The vector $\mathbf{x} = \frac{1}{2}\mathbf{q}$ is a solution to the problem $\min\{\mathbb{1} \cdot \mathbf{x} : \mathbf{x} \in \mathbf{H}\}$, as explained after the proof of Theorem 2.3)

Step 3. Construct a multiflow $\mathbf{q}$-locking $\mathscr{H}$ by applying a locking algorithm to the triple $(G', T', \mathscr{H}')$ where T' is a disjoint copy of T, $\mathscr{H}'$ consists of the copies A', $A \in \mathscr{H}$, and G' is the extension of G by linking each $t \in T$ to its copy $t' \in T'$ by an edge of multiplicity $q(t)$.

(The multiflow constructed at Step 3 solves Problem 0, by Theorem 2.3)

Let us roughly estimate the complexity of the above solution, assuming that the network is given by an underlying graph $G = (V, E)$ with n vertices and m edges and a vector $(c(e) : e \in E)$ of the edge multiplicities.

1. When constructing the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ on Step 1, choose for V_A the inclusion-minimal A-mincut. This may be done in strongly polynomial time, for example, by solving the locking problem for $(G, T, \mathscr{H})$ [9, 3]. Such choice guarantees, by the submodularity inequality (6), that $V_A \cap V_B = \varnothing$ whenever $A \cap B = \varnothing$, so that a vertex of G is duplicated in $\tilde{G}$ at most twice. We have therefore $|\tilde{V}| \leqslant 2n$ and $|\tilde{E}| \leqslant 4m$.

2. It is shown in [7], Theorem 2.3, that the degrees of a maximum balanced flow in $\tilde{G}$ may be determined by maximizing a **b**-matching in a graph with $|\tilde{V}|+|\tilde{E}| \leqslant 2n+4m$ vertices, obtained from $\tilde{G}-s$ by the following two operations: (1) subdividing each edge $e \in \tilde{E}$ by a 2-valent vertex, and (2) appending the pairs of U as additional edges. (In the new graph, the b-value of a vertex is either $\frac{1}{2}d_G(v)$, for a copy of an initial vertex $v \in V$, or $c(e)$, for the vertex subdividing an edge e.) A strongly polynomial matching algorithm (see [5, Theorem 25]) provides the half-degree vector of a maximum Eulerian balanced flow; this is what one actually needs for Step 3.

3. After the demand vector $\mathbf{q}$ is calculated (Step 2), the locking algorithm with the input $(G', T', \mathscr{H}')$ constructs a solution of Problem 0 in time polynomial in n.

Thus, a minimum multiflow locking a given 2-regular discrete hypergraph can be constructed by calling twice to an algorithm of usual locking and once to a **b**-matching algorithm.

Computations are easier in the important particular case when the dual hypergraph $\mathscr{H}^*$, actually a graph in our case, is bipartite (cf. [3]). The latter means that $\mathscr{H}$ admits a partition into two hypergraphs, $\mathscr{H}_1$ and $\mathscr{H}_2$, each consisting of pairwise disjoint sets. In such a case the demand vector $\mathbf{q}$ may be obtained by solving a maximum flow problem in a network similar to that suggested by Karzanov [9]. This network only slightly differs from the spread defined above. Namely, one first constructs the disjoint union of the same subgraphs G_A, in which the vertices s_A, $A \in \mathscr{H}_1$, are identified into one, say s_1, called the *source*, and the vertices s_A, $A \in \mathscr{H}_2$, are identified into another one, s_2, called the *sink*. The pairs of U are then appended as edges of infinite capacity. In the network thus obtained, we construct a maximum Eulerian (s_1, s_2)-flow having even values on the edges from U. These values are then used, in the same way as before, for computing the demands $\mathbf{q}$.

It is shown in [9] that in such a network there exists a maximum flow with even values in the edges of U, and its complexity is, essentially, the same as of the general one.

2.4. *Interpretation of the Sandwich Formula*

The equality (3) of the main theorem is derived here from the sandwich formula of [7].

Let $\tilde{G}$ be an Eulerian graph with the vertex-set $\tilde{V} \cup \{s\}$, and U be a partial pairing of $\tilde{V}$. A pair (X, Y) of disjoint sets of vertices is called a *sandwich* if $s \in X$, and any pair of U with one member in X has the other one in Y. It is proved in [7] that $\max|\mathscr{F}| = \min(d(X)+d(Y)-2\omega)$, the maximum over the Eulerian balanced flows $\mathscr{F}$, and the minimum over the sandwiches (X, Y). Here ω denotes the number of odd components of the subgraph $\tilde{G}+U-(X\cup Y)$, the parity of a component with the vertex-set C being defined as the parity of the integer $\frac{1}{2}(d(C\cup Y)-d(Y))$.

Suppose now that the graph $\tilde{G}$ and the partial pairing U of $\tilde{V}$ are those constructed for a triple $(G, T, \mathscr{H})$ in Subsection 2.3. By Theorem 2.4, we have

$$\Lambda-\sigma=\min\frac{d(X)+d(Y)-2\omega}{4}, \tag{19}$$

and it remains to show that the right-hand side of (19) coincides with the minimum capacity of a kernel 2-cover of $(G, T, \mathscr{H})$. For this purpose we construct the graph $\tilde{G}$ by choosing the maximal subsets V_A; that is, $V_A=V\backslash A^c$, $A\in\mathscr{H}$. Then there is one-to-one correspondence between the kernel 2-covers of $(G, T, \mathscr{H})$ and the sandwiches of $(\tilde{G}, U, s)$.

Indeed, let (X, Y) be a sandwich of $(\tilde{G}, U)$. For each $A\in\mathscr{H}$, form the two sets $K_A:=V(G_A)\cap Y$ and $J_A:=V(G_A)\backslash X$, and consider them as subsets of V_A in the initial graph G. Let us show that the obtained family $\mathscr{W}:=(K_A, J_A: A\in\mathscr{H})$ is a kernel 2-cover. Obviously, we have $K_A\subseteq J_A$ and $J_A\cap T\subseteq A$; it remains to check that each terminal is covered at least twice. Choose $t\in T$, and let A and B be the members of $\mathscr{H}$ containing t (since $\mathscr{H}$ is 2-regular). Denote by t_A and t_B the copies of t in the spread belonging to G_A and G_B respectively. If one of them, say t_A, lies in Y then $t\in K_A\subseteq J_A$, so that $\mathscr{W}$ covers t at least twice. If neither of t_A, t_B lies in Y, then both are in $\tilde{G}-X$, by the definition of sandwich. Then t belongs to J_A and J_B.

Conversely, given a kernel 2-cover $\mathscr{W}$, let, for every $A\in\mathscr{H}$, the members K_A and J_A of $\mathscr{W}$ be considered as subsets of $V(G_A)\subset\tilde{V}$; by the maximality of $\tilde{G}$, this is always possible. Let us form the sets $Y:=\bigcup_{A\in\mathscr{H}}K_A$ and $X:=\tilde{V}\cup\{s\}\backslash\bigcup_{A\in\mathscr{H}}J_A$ and prove that (X, Y) is a sandwich. The relations $s\in X$ and $X\cap Y=\varnothing$ are obvious. Consider a pair $(t_A, t_B)\in U$ formed by the copies of a terminal $t\in T$ in G_A and G_B, respectively. If $t_A\in X$ then $t\notin J_A$. Then the (at least) two members of $\mathscr{W}$ containing t should inevitably be K_B and J_B, because $t\notin J_C$ for all $C\in\mathscr{H}$ distinct from A and B. Thus, $t\in Y$, as required.

Now let a sandwich (X, Y) of $(\tilde{G}, U, s)$ and a kernel 2-cover $\mathscr{W}$ of $(G, T, \mathscr{H})$ correspond to each other in the above way. Then, since the subgraphs G_A are isolated in $\tilde{G}-s$, we have $d(X)=d(\tilde{V}\backslash X)=\sum_{A\in\mathscr{H}}d(J_A)$ and $d(Y)=\sum_{A\in\mathscr{H}}d(K_A)$, whence $d(X)+d(Y)=\sum_{Z\in\mathscr{W}}d(Z)$.

It is, further, easily seen that the components of $\tilde{G}+U-(X\cup Y)$ correspond, in the obvious sense, to the $\mathscr{W}$-components, and that the definition of the component parity is the same in both cases.

This completes the proof of the main theorem.

3. THE INNER EULERIAN CASE

An inner Eulerian network can be made Eulerian by appending a new terminal and linking it to the terminals of odd degree. This construction, together with the main theorem, immediately provides a characterization of σ for inner Eulerian networks; some preparations are, however, needed if one wants the present form of the main theorem to be preserved. We start with the following refinement of the main theorem. Let us call a kernel 2-cover $\mathscr{W}$ *strict* if each terminal belongs to *exactly* two members of $\mathscr{W}$.

CLAIM 3.1. *Let G be Eulerian and $\mathscr{H}$ be 2-regular and discrete. Then there exists a minimum kernel 2-cover which is strict.*

Proof. We construct a sequence $\mathscr{W}_n$ of minimum kernel 2-covers for $(G, T, \mathscr{H})$, which becomes strict for n large enough.

Add a new terminal, u, and form a new graph G' by linking u to the members of T, to each by $4n$ parallel edges. Thus, $T' := T \cup \{u\}$ is the new terminal-set, and let $\mathscr{H}'$ consist of $\mathscr{H}$ and two copies of the singleton $\{u\}$. We use primes to distinguish parameters related to $(G', T', \mathscr{H}')$, such as σ', $d'(X)$, $\|\mathscr{W}'\|'$, *etc*., from their counterparts σ, $d(X)$, $\|\mathscr{W}\|$, *etc*., related to $(G, T, \mathscr{H})$.

There obviously holds $\sigma' = \sigma + 4n\,|T|$. Further, for a subset $X \subseteq V(G)$ we have $d'(X) = d(X) + 4n\,|T \cap X|$, whence $\lambda'(A) = \lambda(A) + 4n\,|A|$, and

$$\Lambda' = \Lambda + \tfrac{1}{2}\left(4n \sum_{A \in \mathscr{H}} |A| + 2d'(u)\right) = \Lambda + 8n\,|T|,$$

because $\mathscr{H}$ is 2-regular.

Let, on the other hand, $\mathscr{W}'$ be a minimum kernel 2-cover for the triple $(G', T', \mathscr{H}')$, and $\mathscr{W}_n$ denote its part consisting of subsets of $V(G)$, so that $\mathscr{W}' = \mathscr{W}_n \cup \{K_u, J_u\}$ where $\emptyset \subseteq K_u \subseteq J_u$ and $J_u \cap T' = \{u\}$. Since $T \cap J_u = \emptyset$, the family $\mathscr{W}_n$ is a kernel 2-cover for the initial triple $(G, T, \mathscr{H})$. Since, further, u should be covered twice, we have $u \in K_u$, which implies that $\mathscr{W}'$ and $\mathscr{W}_n$ have exactly the same components. Moreover, they have the same odd components, because, by the formula (4), the new parity parameter $p'(\mathscr{C})$ differs from the initial value $p(\mathscr{C})$ by a multiple of $2n$.

Let $m_n(t)$ denote the number of sets of $\mathscr{W}_n$ containing a terminal $t \in T$. By the definition of a kernel 2-cover, $m_n(t) \geqslant 2$; let us show that $m_n(t) = 2$ when n is large enough. We have

$$\sum_{X \in \mathscr{W}'} d'(X) \geqslant \sum_{X \in \mathscr{W}_n} (d(X) + 4n\,|T \cap X|) + 8n\,|T|$$

$$= \sum_{X \in \mathscr{W}_n} d(X) + 4n \sum_{t \in T} (m_n(t) + 2), \tag{20}$$

whence, by the main theorem,

$$\Lambda - \sigma = \Lambda' - \sigma' - 4n\,|T| = \|\mathscr{W}'\|' - 4n\,|T|$$

$$\geqslant \frac{1}{4} \sum_{X \in \mathscr{W}_n} d(X) + n \sum_{t \in T} (m_n(t) - 2)) - \frac{\omega'}{2} \tag{21}$$

$$= \|\mathscr{W}_n\| + n \sum_{t \in T} (m_n(t) - 2)) \tag{22}$$

because ω' coincides with the number of odd $\mathscr{W}_n$-components. Since both $\Lambda - \sigma$ and $\|\mathscr{W}_n\|$ are related to the initial graph and are therefore bounded, we have $m_n(t) = 2$ for all T (whence $\mathscr{W}_n$ is strict) when n is large enough. Thus, $\Lambda - \sigma = \|\mathscr{W}_n\|$; by the main theorem, this implies that the kernel 2-cover $\mathscr{W}_n$ is minimum. ∎

Suppose now that (G, T) is inner Eulerian, and let M denote the set of odd terminals. We make the graph Eulerian by appending a new terminal, u, and linking it by an edge to each $t \in M$. Again, $T' := T \cup \{u\}$ is the new terminal-set, and $\mathscr{H}'$ is the hypergraph on T' consisting of $\mathscr{H}$ and two copies of $\{u\}$. For the triple $(G', T', \mathscr{H}')$ we have $\sigma' = \sigma + |M|$ and

$$\Lambda' = \tfrac{1}{2}\left(\sum_{A \in \mathscr{H}} (\lambda(A) + |A \cap M|) + 2\,|M| \right) = \Lambda + 2\,|M|,$$

because $\mathscr{H}$ is 2-regular. Thus, $\Lambda' - \sigma' = \Lambda - \sigma + |M|$.

Let $\mathscr{W}'$ be a strict minimum kernel 2-cover for $(G;, T;, \mathscr{H}')$, by Claim 3.1. As before, it consists of a strict kernel 2-cover $\mathscr{W}$ for $(G, T, \mathscr{H})$ and sets K_u and J_u satisfying $u \in K_u \subseteq J_u$, $T \cap J_u = \emptyset$. It is easy to check that the minimum of $\|\mathscr{W}'\|'$ is achievable with the choice $K_u = J_u = \{u\}$. (Indeed, since K_u, J_u contain no other terminal, we have $d'(K_u), d'(J_u) \geqslant |M| = d'(u)$; for the same reason, the choice of K_u, J_u does not affect the $\mathscr{W}'$-components.) Since G' is Eulerian, we apply the main theorem to obtain

$$\Lambda' - \sigma' = \|\mathscr{W}'\|' = \frac{1}{4}\left(\sum_{X \in \mathscr{W}} (d(X) + |M \cap X|) + 2\,|M| \right) - \frac{\omega'}{2}$$

$$= \frac{1}{4} \sum_{X \in \mathscr{W}} d(X) + |M| - \frac{\omega'}{2}, \tag{23}$$

where the latter equality follows from the assumption that each terminal is covered by $\mathscr{W}'$ exactly twice. So, for inner Eulerian networks the main theorem has the modified form

$$\Lambda - \sigma = \Lambda' - \sigma' - |M| = \min\left(\frac{1}{4}\sum_{X\in\mathscr{W}} d(X) - \frac{\omega'}{2}\right), \tag{24}$$

the minimum over the strict kernel 2-covers. Again, we observe that $\mathscr{W}'$ and $\mathscr{W}$ have exactly the same components, so that the equality (24) simply means that the $\mathscr{W}$-component parity should now be defined as for $\mathscr{W}'$-components, that is, in terms of the modified parameter $p'(\mathscr{C}) = \frac{1}{2}\sum_{X\in\mathscr{C}}\delta'(X)$. One easily sees that $\delta'(X) = \delta(X) + |M\cap X|$, so that, denoting by $M(\mathscr{C})$ the set of odd terminals in $\mathscr{C}$, we may generalize the parity parameter by setting

$$p(\mathscr{C}) := \sum_{X\in\mathscr{C}} \delta(X) + |M(\mathscr{C})| \qquad (= p'(\mathscr{C})). \tag{25}$$

In terms of the new parity parameter (25), the form of the main theorem remains almost unchanged:

THEOREM 3.2. *If G is inner Eulerian and $\mathscr{H}$ is 2-regular and discrete then*

$$\sigma = \Lambda - \min\|\mathscr{W}\|,$$

the minimum over all strict kernel 2-covers. Here

$$\|\mathscr{W}\| = \frac{1}{2}\sum_{A\in\mathscr{H}} d(K_A) + \sum_{\mathscr{C}}\left\lfloor\frac{1}{2}p(\mathscr{C})\right\rfloor = \frac{1}{4}\sum_{X\in\mathscr{W}} d(X) - \frac{\omega}{2},$$

where $\mathscr{C}$ denotes a $\mathscr{W}$-component and ω is the number of odd $\mathscr{W}$-components.

4. APPLICATION TO PACKING S-PATHS

Given a network (G, T) and a simple graph S on the vertex-set T (called *scheme*), a T-path will be called an S-*path* if its ends are adjacent in S. A set of edge-disjoint S-paths will be called an S-*flow*. The scheme will always be assumed to have no isolated vertices. In this section we deal with packing edge-disjoint S-paths (in other words, with maximization of an S-flow) for the schemes satisfying the condition that each terminal belongs to at most two anticliques (maximal stable sets). In [12] such schemes are called *loose*. We solve this problem for inner-Eulerian networks by applying the above result on the minimum locking.

The maximum cardinality of an S-flow in the network will be denoted by θ. The loose graphs admit the following simple description.

CLAIM 4.1. *Let S be a simple graph without isolated vertices and T be its vertex-set. The following statements are equivalent.*

(4.1.1) *S is loose*;

(4.1.2) *its complement $\bar{S}$ is the line graph of a triangle-free graph*;

(4.1.3) *there exists a 2-regular discrete hypergraph $\mathscr{H}$ on T such that two terminals are adjacent in S iff no set $A \in \mathscr{H}$ contains both of them.*

As usual, the term graph includes multigraphs. It will be seen in the forthcoming proof that the triangle-free graph of (4.1.2) is actually the dual $\mathscr{H}^*$ of the hypergraph mentioned in (4.1.3) (cf. Claim 1.2).

Proof of Claim 4.1.

(4.1.3) $\to$ (4.1.2). We show that if S satisfies (4.1.3) then $\bar{S}$ is the line graph of $\mathscr{H}^*$, which, by Claim 1.2, has no triangle. Indeed, $t_1, t_2 \in T$ are adjacent in $\bar{S}$ if and only if $t_1, t_2 \in A$ for some set $A \in \mathscr{H}$ or, in other words, if and only if the edges t_1 and t_2 of the graph $\mathscr{H}^*$ have a common end.

(4.1.2) $\to$ (4.1.1). Let $\bar{S}$ be the line graph of some triangle-free multigraph, J. Since J has no triangle, the cliques of $\bar{S}$ are just the inclusion-maximal stars of J, and it is then clear that each vertex of $\bar{S}$ (i.e., an edge of J) belongs to at most two cliques. (The star of a vertex v is a subset of the star of another vertex, u, iff v is linked to no other vertex except u; then an edge between v and u belongs to a single clique of $\bar{S}$, namely, the star of u in the graph J.)

(4.1.1) $\to$ (4.1.3). Let $\mathscr{A}$ be the collection of anticliques of S, and for each $A \in \mathscr{A}$ let R_A denote the set of terminals in A belonging to no other anticlique. We construct a 2-regular hypergraph $\mathscr{H}$ by appending to $\mathscr{A}$ the nonempty subsets R_A. To show that $\mathscr{H}$ is discrete, consider some set $A \in \mathscr{H}$ and a terminal $t \in A^c$. Let B_1 and B_2 be the two members of $\mathscr{H}$ containing t, and suppose that both meet A. Then B_1, B_2, and A are anticliques. Choose $t_i \in A \cap B_i$, $i = 1, 2$. Since the terminals t_1, t_2, and t are pairwise nonadjacent, they belong to some anticlique, C. Since S is loose, $t_1 \notin B_2$ whence $C \neq B_2$; similarly, $C \neq B_1$. Thus, t belongs to three distinct anticliques of S, contradiction. Thus, $\mathscr{H}$ is 2-regular and discrete, and two terminals are adjacent in S iff they are not covered by the same member of $\mathscr{H}$. ∎

In what follows, packing S-paths is dealt with in terms of the hypergraph $\mathscr{H}$ generating the scheme according to Claim 4.1.

The following characterization of θ is an almost immediate consequence of the main theorem. To prove it, we are only to show that a multiflow, which locks $\mathscr{H}$ and has the minimum cardinality, contains θ S-paths. By a different method, Theorem 4.2 has first been proved in [6].

THEOREM 4.2. *If the network (G, T) is inner Eulerian and the scheme S is loose then*

$$\theta = \min \|\mathscr{W}\|, \tag{26}$$

the minimum over the strict kernel 2-covers. If the graph G is Eulerian then the word strict may be omitted.

Before proving the theorem, it might be interesting to explain in a few words the origin of the notion of a loose scheme. It arose in studying the fractional version of packing S-paths [10, 8, 11, 12] to describe the cases when the solution is representable in terms of saturating certain cuts. A *fractional S-flow* is a nonnegative real weight function $f(P)$ defined on the S-paths of the given network and satisfying the unity capacity constraints on the edges; the *size* $\|f\|$ of an S-flow is, by definition, the total weight of all S-paths. Put $\tilde{\theta} := \sup \|f\|$, over all fractional S-flows f in G; then, clearly, $\theta \leqslant \tilde{\theta}$.

Let, on the other hand, a subset $X \subseteq V$ be called (u, v)-cut, for some vertices u, v, if X contains exactly one of them. Consider weight functions $g : 2^V \to \mathbb{R}_+$, and define $\|g\| := \sum_{X \subseteq V} g(X)\, d(X)$. Let $m_g(u, v)$ denote the total weight of the (u, v)-cuts, for every $u, v \in V$. One easily checks that m_g is a distance function on V. If the weight function g satisfies the distance constraints

$$m_g(t_1, t_2) \geqslant 1, \qquad \text{for any terminals } t_1, t_2 \text{ adjacent in } S, \tag{27}$$

then the inequality $\|f\| \leqslant \|g\|$ holds for any fractional S-flow. In fact, the functions g satisfying (27) form a proper subclass of dual feasible vectors for the fractional S-flow problem. It turns out [10, 8, 12, 11] that the equality $\tilde{\theta} = \min \|g\|$, over the set of such functions, holds for a *given* scheme S and *every* network (G, T) if and only if S is loose. Moreover, in this case a *half-integer* maximum S-flow always exists, provided the network is inner Eulerian.

Now let g be a nonnegative set-function g satisfy the distance constraints (27) and also the equality $\|g\| = \tilde{\theta}$. Then every cut X having $g(X) > 0$ is *saturated* by any maximum S-flow f, in the sense that

(1) each edge of the cut is saturated by f (meaning that the capacity constraint is satisfied with equality), and

(2) every path P having $f(P) > 0$ has at most one edge in common with the cut. In other words, the paths having a common edge with X form

a maximum (A, A^c)-flow, for $A = T \cap X$. (An edge belongs to the cut if it has exactly one end in X.)

The loose schemes guaranteeing the equality $\theta = \tilde{\theta}$ for every inner Eulerian network are exactly those whose graph $\mathscr{H}^*$ is bipartite [10] (see also [2, 3]); Frank, Karzanov and Sebő call such schemes *bi-stable*. Thus, as far as inner Eulerian networks are dealt with, the bi-stable S-paths packing problem resembles the edge-disjoint Menger problem and its relation to the max-flow min-cut theorem: first, the solution is expressed in terms of cuts saturation, and second, the integrality is not an actual constraint but a property of the fractional version of the problem. Frank, *et al.* [2, 3] explained this phenomenon of bi-stable schemes by deriving it from the Edmonds polymatroid intersection theorem and the locking theorem (see Introduction, Theorem 1.1).

Theorem 4.2 deals with a different situation, where the fractional paths packing may have no integer solution. The matter as a whole resembles matchings in general graphs versus matchings in bipartite graphs, and we have already seen that graph matchings have indeed much to do with Problem 0. In an earlier work [6], inspired by Frank, Karzanov and Sebő's insight, formula (26) is derived from the locking theorem and the Lovász polymatroid matching theorem [13]. Later, after the polymatroid at hand was found to be dually representable by bipartite b-matchings (see [7, Theorem 2.3]), we simplified the proof by translating the S-paths packing into b-matching maximization in a graph (see the preceeding sections). In fact, the balanced flow maximization [7] and minimum locking have been posed as auxiliary problem, for constructing a purely graphic framework for this translation.

It should be emphasized that the results of this paper are confined to the inner Eulerian networks, and apparently imply no generalization to the Mader theorem [14] on T-paths packing in arbitrary graphs.

Proof of Theorem 4.2. We start with establishing the following important fact.

Claim 4.3. *There exists a multiflow which locks $\mathscr{H}$ and contains θ S-paths.*

Proof. Given a T-flow $\mathscr{F}$, let n_A denote the number of (A, A^c)-paths in $\mathscr{F}$, and put $\alpha := \frac{1}{2}\sum_{A \in \mathscr{H}} n_A$. Clearly, $\mathscr{F}$ locks $\mathscr{H}$ iff $\alpha = \Lambda$ (see Introduction). Let $\mathscr{F}$ be a T-flow containing θ S-paths and having the largest value of α consistent with that. Then $\mathscr{F}$ locks $\mathscr{H}$, by Lemma 5.2. ∎

By Claim 4.3, it suffices to maximize the number of S-paths in a multiflow over the multiflows locking $\mathscr{H}$; Theorem 4.2 is then obtained as a consequence of the main theorem and the following result.

THEOREM 4.4. *Let $\mathscr{H}$ be a 2-regular discrete hypergraph, $\mathscr{H}^*$ be its dual, and S be the complement of the line graph of $\mathscr{H}^*$. Then any solution of Problem 0 for $(G, T, \mathscr{H})$ contains θ S-paths. Moreover,*

$$\theta + \sigma = \Lambda. \tag{28}$$

Proof. Let $\beta_k(\mathscr{F})$ denote the number of paths of $\mathscr{F}$ whose pair of ends belongs to exactly k members of $\mathscr{H}$. In particular, $\beta_0(\mathscr{F})$ is the number of S-paths in $\mathscr{F}$, so that $\theta = \max \beta_0(\mathscr{F})$ over the multiflows locking $\mathscr{H}$.

Let $\mathscr{F}_A$ denote the set of A-paths in $\mathscr{F}$. Then $\sum_{A \in \mathscr{H}} |\mathscr{F}_A| = \beta_1(\mathscr{F}) + 2\beta_2(\mathscr{F})$, whence

$$\begin{aligned} 2\,|\mathscr{F}| &= \sum_{t \in T} d_{\mathscr{F}}(t) = \tfrac{1}{2} \sum_{A \in \mathscr{H}} \sum_{t \in A} d_{\mathscr{F}}(t), && \text{since } \mathscr{H} \text{ is 2-regular} \\ &= \tfrac{1}{2} \sum_{A \in \mathscr{H}} (\lambda(A) + 2\,|\mathscr{F}_A|), && \text{because } \mathscr{F} \text{ locks each } A \in \mathscr{H}, \\ &= \Lambda + \beta_1(\mathscr{F}) + 2\beta_2(\mathscr{F}). \end{aligned} \tag{29}$$

Eliminating $|\mathscr{F}|$ from the equalities (29) and $|\mathscr{F}| = \beta_0(\mathscr{F}) + \beta_1(\mathscr{F}) + \beta_2(\mathscr{F})$, we obtain the relation

$$2\beta_0(\mathscr{F}) = \Lambda - \beta_1(\mathscr{F}), \tag{30}$$

which holds when $\mathscr{F}$ locks $\mathscr{H}$. It remains to show that the minimization of $|\mathscr{F}|$ over such multiflows minimizes also the number $\beta_1(\mathscr{F})$. This is an immediate consequence of (29) and the following trivial fact.

CLAIM 4.5. *For any multiflow $\mathscr{F}'$ locking $\mathscr{H}$ there is a multiflow $\mathscr{F}$ with $\beta_2(\mathscr{F}) = 0$ which also locks $\mathscr{H}$ and has the same β_0 and β_1.*

Proof. It suffices to show that a path $P \in \mathscr{F}'$ whose ends belong to $A \cap B$ for some distinct $A, B \in \mathscr{H}$ can be removed. Indeed, since $\mathscr{H}$ is 2-regular, the ends of P are not separated by any member of $\mathscr{H}$. Since $\mathscr{H}$ is locked, P has no edge in common with any minimum C-cut, $C \in \mathscr{H}$. Therefore $\mathscr{F}' \backslash \{P\}$ still locks $\mathscr{H}$; clearly, it has the same numbers β_0 and β_1. ∎

Returning to Theorem 4.4, we notice that every solution $\mathscr{F}$ of Problem 0 minimizes $\beta_1(\mathscr{F})$ over the multiflows locking $\mathscr{H}$ and therefore has $\beta_0(\mathscr{F}) = \max$, by (30). Claim 4.3 asserts that this maximum equals θ. Moreover, we have $2\sigma = \Lambda + \min \beta_1(\mathscr{F})$, by (29) and Claim 4.5. Since we also have $2\theta = \Lambda - \min \beta_1(\mathscr{F})$, by (30), the equality (28) follows. ∎

5. APPENDIX: TWO LEMMAS ON MULTIFLOWS

We prove here two technical lemmas used in the main text.

I. Given a nonnegative vector $\mathbf{w} = (w(t) : t \in T)$, the sum $\mathbf{w} \cdot \mathbf{d}_{\mathscr{F}} = \sum_{t \in T} w(t)\, d_{\mathscr{F}}(t)$ will be called the *weight* of a T-flow $\mathscr{F}$.

LEMMA 5.1. *Let G be an Eulerian graph, $T \subseteq V(G)$, $\mathscr{H}$ be an arbitrary hypergraph on T, and $\mathbf{w} \in \mathbb{Z}_+^T$. Then for any multiflow locking $\mathscr{H}$ there exists an Eulerian multiflow of the same or smaller weight which also locks $\mathscr{H}$.*

Proof. Let a multiflow $\mathscr{F} = \{P_1, ..., P_m\}$ lock $\mathscr{H}$, and let $t_1, t_2, ..., t_{2k-1}, t_{2k}$ be the terminals in which $d_{\mathscr{F}}(t)$ is odd. Since G is Eulerian, there are paths $Q_1, ..., Q_k$ such that $P_1, ..., P_m, Q_1, ..., Q_k$ are edge-disjoint and have the joint degree even everywhere. Without loss of generality, let the ends of Q_j be t_{2j-1} and t_{2j}, and $w(t_{2j}) \geqslant w(t_{2j-1})$, $j = 1, ..., k$. Since $\mathscr{F}$ locks $\mathscr{H}$, no Q_j meets any minimum A-cut, $A \in \mathscr{H}$. In order to construct the required multiflow, choose for each $j = 1, ..., k$ a path $P_{i_j} \in \mathscr{F}$ having an end in t_{2j} (such a path exists because $d_{\mathscr{F}}(t_{2j})$ is odd), and concatenate it with Q_j in this end. (If some P_i, with the ends t_{2r} and t_{2s}, is chosen twice, merge it with both Q_r and Q_s.)

The new multiflow is Eulerian and has at most the same weight due to the choice of P_{i_j}. Clearly, it locks $\mathscr{H}$ because every minimum A-cut, $A \in \mathscr{H}$, remains saturated, and the new paths have at most one edge in common with it. ∎

II. Let us now slightly extend the notion of a T-flow by permitting paths to be self-intersecting, closed, and degenerate. By the latter we mean a path consisting of a single terminal, and in what follows we assume that the degenerate paths $\{t\}$, $t \in T$, are always present. Clearly, a T-flow in the new sense locks a subset of T iff it does so after removing self-intersections and deleting closed and degenerate paths.

Given a T-flow $\mathscr{F}$ in a network (G, T), let us consider two parameters: the number β of S-paths in $\mathscr{F}$ and $\alpha := \frac{1}{2} \sum_{A \in \mathscr{H}} n_A$ where n_A is the number of (A, A^c)-paths in $\mathscr{F}$. Clearly, α attains its maximum (equal to Λ) if and only if $\mathscr{F}$ locks $\mathscr{H}$. Our aim is to transform a T-flow $\mathscr{F}$ which does not lock $\mathscr{H}$ into another one having greater α and at least the same β. The construction is based on the familiar notion of augmenting path and uses a transformation of a multiflow which we call *switch* (cf. [12]).

LEMMA 5.2. *Let G be inner Eulerian and $\mathscr{H}$ be 2-regular and 3-cross free. If a T-flow $\mathscr{F}$ does not lock $\mathscr{H}$ then there exists another T-flow, with greater α and at least the same β.*

We prove the lemma after some necessary preliminaries. The following criterion is essentially the well-known augmenting path theorem for network flows. Let $\mathscr{F}$ be a T-flow and A be a proper subset of T. Let D denote the partial orientation of G obtained by directing the (A, A^c)-paths of $\mathscr{F}$ towards A and choosing an arbitrary Eulerian orientation of the subgraph consisting of the free edges. Finally, let X and Y be the sets of vertices lying on the A-paths and the A^c-paths of $\mathscr{F}$, respectively. Then the following alternative takes place [12].

CLAIM 5.3. *$\mathscr{F}$ does not lock A if and only if there exists a directed (X, Y)-path in D.*

In [12] the locking problem was solved by implementing an augmenting path as a sequence of switches. Now, we additionally check how such a transformation affects the number of S-paths in the multiflow.

Given a 2-regular discrete hypergraph $\mathscr{H}$ on T, let us assign to every $p, q \in T$ the *weight* $w(p, q)$ equal to the number of sets $A \in \mathscr{H}$ satisfying $p, q \in A$. Clearly, we always have $w(p, q) \leqslant 2$ and $w(p, q) = 0$ iff p and q are adjacent in S. The equation $w(p, q) = 2$ defines an equivalence on T; for equivalent p, q the equality $w(p, t) = w(q, t)$ holds for any $t \in T$. Finally, let the weight of a T-path P be defined by $w(P) := w(p, q)$ where p, q are the ends of P.

Consider two edge-disjoint T-paths, P and Q, with a common vertex, v. The *switch* is a transformation of P and Q into two other such paths, R_1 and R_2, having the same joint edge-set and mixed end-pairs, coupling an end of P with an end of Q, in one of the two possible ways. Let the ends of Q be denoted by q_1 and q_2, and we assume that R_i is that of resulting paths which contains q_i. After a realization of switch is chosen, the end of P belonging to R_i will be denoted by p_i. In these notations, the resulting paths have the standard form

$$R_1 = p_1 P v Q q_1 \qquad \text{and} \qquad R_2 = p_2 P v Q q_2 \tag{31}$$

(where $pPvQq$ means concatenating the segment of P, between the vertices p and v, with the segment of Q between v and q). Thus, the switch is actually fixed by indicating the ends of P. The two ways of doing this are equivalent when $w(P) = 2$; in the other cases we choose a switch so as to minimize the value of

$$\max\{w(R_1), w(R_2)\} \tag{32}$$

and call such a choice *optimal*.

For our purposes, P will always be chosen to have both ends in some set $A \in \mathscr{H}$ (so that $w(P) \geqslant 1$), while at least one end of Q, say q_2, will be in A^c. The set of $\mathscr{H}$ containing p_i and distinct from A will be denoted by B_i.

The absence of 3-semicrosses in $\mathscr{H}$ enables further standardization, as follows.

CLAIM 5.4. *If $w(P)=1$ then an optimal switch can be chosen so that either $q_1, q_2 \in B_2$ or $q_i \notin B_i$, $i=1, 2$.*

Proof. The condition $w(P)=1$ means that $B_1 \neq B_2$: we have then $B_1 \cap B_2 = \emptyset$, for otherwise B_1, B_2 and A form a 3-semicross. Thus, each end of Q belongs to at most one set B_i.

Suppose first that this B_i is the same for q_1 and q_2; then the assignment $i := 2$ is optimal. Indeed, we have then $w(R_2) = w(p_2, q_2) = 1$, because $q_2 \notin A$, and $w(R_1) = w(p_1, q_1) = 1$, because $q_1 \notin B_1$. The alternative assignment $i := 1$ provides $p_1, q_1 \in B_1$, and thereby $w(p_1, q_1) \geqslant 1$; the former one is therefore optimal.

Otherwise, there are distinct i_1 and i_2 such that $q_1 \notin B_{i_1}$ and $q_2 \notin B_{i_2}$. Then the assignment $i_1 := 1$, $i_2 := 2$ is optimal. Indeed, this is obvious when $q_1 \notin A$, because then R_1, R_2 are S-paths. In the case $q_1 \in A$ we have $w(R_1) = 1$ and $w(R_2) = 0$. Since, however, $w(R_1) \geqslant 1$ holds for the alternative switch too, the former one is optimal. ▮

Let us now consider $\{P, Q\}$ and $\{R_1, R_2\}$ as multiflows and evaluate the increments $\Delta\alpha = \alpha(R_1, R_2) - \alpha(P, Q)$ and $\Delta\beta = \beta(R_1, R_2) - \beta(P, Q)$. Since P is never an S-path, we have $\beta(P, Q) \leqslant 1$ whence $\Delta\beta \geqslant -1$. There are four cases to look at, depending on whether $w(P) = 1$ or 2 and whether q_1 lies in A or in A^c. We summarize them in the following

CLAIM 5.5. *Let an optimal switch be chosen according to Claim* 5.4. *Then*

(5.5.1) *There always holds $\Delta\alpha \geqslant 0$, and if $q_1 \notin A$ then $\Delta\alpha > 0$.*

(5.5.2) *$\Delta\beta < 0$ occurs only when $B_1 = B_2 (= B)$, $q_1 \in A \backslash B$, and $q_2 \in B \backslash A$; in this case $\Delta\beta = -1$, and R_1 is an A-path of weight* 1.

(5.5.3) *Suppose that $w(P) = 1$. If $q_1 \in A$ then R_1 is an A-path of weight* 1; *if $q_1 \notin A$ then $\Delta\beta > 0$.*

Proof. (1) Let us first show that $\Delta\alpha \geqslant \Delta n_A$. Indeed, we have

$$\Delta\alpha \geqslant \Delta n_A + \begin{cases} \Delta n_{B_1} + \Delta n_{B_2}, & \text{if} \quad B_1 \neq B_2 \\ \Delta n_B, & \text{if} \quad B_1 = B_2 (= B). \end{cases} \tag{33}$$

If $B_1 = B_2$ then we clearly have $\Delta n_B \geqslant 0$ because P is also a B-path.

If $B_1 \neq B_2$ and $q_1, q_2 \in B_2$ then $\Delta n_{B_2} \geqslant 0$, because Q is a B_2-path, and $\Delta n_{B_1} \geqslant 0$, because neither of R_i is a B_1-path.

Finally, if $B_1 \neq B_2$ and $q_i \notin B_i$, $i = 1, 2$ then Δn_{B_1}, $\Delta n_{B_2} = 0$ because neither of the resulting paths is a B_1- or B_2-path.

Thus, $\Delta\alpha \geqslant \Delta n_A$. The assertion (5.5.1) follows from the fact that $\Delta n_A > 0$ when $q_1 \notin A$ and is zero otherwise.

(2) $\Delta\beta < 0$ means that Q is an S-path (i.e., $w(Q) = 0$) while $w(R_i) \geqslant 1$, $i = 1, 2$.

The case $B_1 \neq B_2$ cannot take place because the condition $q_1, q_2 \in B_2$ contradicts the equality $w(Q) = 0$, and the condition $q_i \notin B_i$ contradicts the relation $w(R_i) \geqslant 1$. Thus, there should be $B_1 = B_2 (= B)$. Then the inequalities $w(R_i) \geqslant 1$, $i = 1, 2$, imply $q_1, q_2 \in A \cup B$, while $w(Q) = 0$, together with $q_2 \notin A$, implies that $q_1 \in A \backslash B$ and $q_2 \in B \backslash A$. Therefore R_1 is an A-path, and we have $w(R_1) = w(p_1, q_1) = 1$.

(3) If $w(P) = 1$ (i.e., $B_1 \neq B_2$) then from Claim 5.4 it follows that $q_1 \notin B_1$. Therefore, in the case $q_1 \in A$ the A-path R_1 has $w(R_1) = w(p_1, q_1) = 1$. Further, if $q_1 \notin A$ then $w(R_1) = 0$, because q_1 belongs to neither A nor B_1. This is enough for the inequality $\Delta\beta > 0$ to hold when $w(Q) \neq 0$. It holds also when $w(Q) = 0$, because this is only possible if $q_i \notin B_i$, $i = 1, 2$. Since $q_2 \notin A \cup B_2$, we have $w(R_2) = 0$, whence $\Delta\beta > 0$. ▌

Proof of Lemma 5.2. We may assume, without loss of generality, that $\mathscr{F}$ is inclusion-maximal; that is, G has no T-path (in the extended sense) consisting of free edges (i.e., edges not used by $\mathscr{F}$). Then the subgraph of free edges is Eulerian and contains no terminal.

Let A be a member of $\mathscr{H}$ unlocked by $\mathscr{F}$ and $L = (v_0, v_1, ..., v_n)$ be an augmenting path, by Claim 31. Let P be an A-path passing through v_0. We prove the lemma by interpreting L as a sequence of switch transformations over $\mathscr{F}$ and using induction in n. Since a switch transformation may decrease β (see the case (5.5.2)), the assertion of Lemma should be refined as follows.

Refined assertion: under the assumptions of Lemma 5.2, *there exists a multiflow with greater* α *and at least the same* β; *its* β *is strictly greater when* $w(P) = 1$.

This is true for $n = 0$. Indeed, we have then $v_0 \in Y$, so that there is an A^c-path $Q \in \mathscr{F}$ passing through v_0, with the ends $q_1, q_2 \notin A$. Let us optimally switch P and Q and apply Claim 5.5. Since no end of Q is in A, we have $\Delta\alpha > 0$ by (5.5.1) and $\Delta\beta \geqslant 0$ by (5.5.2). In the case $w(P) = 1$ the inequality $\Delta\beta > 0$ follows from (5.5.3).

Suppose now that $n > 0$, and consider the edge $e = (v_0, v_1)$ of L. If e is free then it belongs to a circuit, C, consisting of free edges, so that $P \cup C$ is an A-path with the same ends as P, passing through v_1. Let $j := \max\{i : v_i \in C\}$. We have $j \geqslant 1$, so that $(v_j, v_{j+1}, ..., v_n)$ is a shorter augmenting path. The refined assertion holds by the induction hypothesis.

Suppose now that e belongs to an (A, A^c)-path $Q \in \mathscr{F}$. We denote its ends by q_1 and q_2 in the way that $q_1 \in A$, $q_2 \in A^c$; by the definition of an augmenting path, the vertices q_1, v_1, v_0, q_2 lie on Q in this order. Again, we optimally switch P and Q and apply Claim 5.5. By (5.5.1), we have $\Delta\alpha \geqslant 0$. Further, R_1 is an A-path passing through v_1, so that there again exists a shorter augmenting path. When applying the induction hypothesis, two possible outcomes of the switch are to be distinguished.

The case $\Delta\beta \geqslant 0$. For $w(P) = 2$, the refined assertion follows directly. If $w(P) = 1$ then also $w(R_1) = 1$, by (5.5.3), so that β eventually grows, by the induction hypothesis.

The case $\Delta\beta < 0$. This is just the point for which the refinement of the lemma is needed. We have here $w(P) = 2$, $\Delta\beta = -1$, and $w(R_1) = 1$, by (5.5.2). Then, by the induction hypothesis, the shorter augmenting path provides the final multiflow having more than $\beta - 1$ (i.e., at least β) S-paths, just enough for the case $w(P) = 2$. ∎

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REFERENCES

1. A. Frank, Packing paths, circuits and cuts—a survey, *in* "Paths, Flows and VLSI-Layout" (B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver, Eds.), Algorithms and Combinatorics, Vol. 9, Springer-Verlag, Berlin, 1990.
2. A. Frank, A. Karzanov, and A. Sebő, On multiflow problems, *in* "Integer Programming and Combinatorial Optimization" (E. Balas, G. Cornuejols, and R. Kannan, Eds.), Vol. 2, pp. 185–101, Carnegie-Mellon Univ., 1992.
3. A. Frank, A. Karzanov, and A. Sebő, On integer multiflow maximization, *SIAM J. Discrete Math.* **10**, No. 1 (1997), 158–170.
4. D. Gale, A theorem on flows in networks, *Pacific J. Math.* **7** (1957), 1073–1082.
5. A. M. H. Gerards, Matching, *in* "Handbooks in OR and MS" (M. O. Ball *et al.*, Eds.), Vol. 7, pp. 135–224, 1995.

6. H. Ilani, E. Korach, and M. Lomonosov, “A New Max-min Theorem on Packing Paths,” Rapport de recherche, Labo. Leibniz, Imag, **RR 970-I**, February 1997.
7. H. Ilani and M. Lomonosov, Maximum balanced flow in a network, *J. Combin. Theory Ser. B* **78** (2000), 185–197.
8. A. Karzanov, (1979). Combinatorial methods to solve cut-determined multiflow problems, *in* “Combinatorial Methods for Flow Problems” (A. Karzanov, Ed.), Issue 3, pp. 6–69, Institute for Systems Studies, Moscow. [In Russian]
9. A. Karzanov, On multicommodity flow problems with integer-valued optimal solutions, *Dokl. Akad. Nauk SSSR* **280** (1985), 789–792. [In Russian, English translation: *Soviet Math. Dokl.* **32** (1985), 151–154]
10. A. Karzanov and M. Lomonosov, (1978). Systems of flows in undirected networks, *in* “Mathematical Programming” (O. I. Larichev, Ed.), Issue 1, Institute for Systems studies, Moscow. [in Russian]
11. A. Karzanov and P. Pevzner, On characterization of the class of cut-determined maximum multicommodity flow problems, *in* “Combinatorial Methods for Flow Problems” (A. Karzanov, Ed.), issue 3, pp. 70–81, Institute for Systems Studies, Moscow. [in Russian]
12. M. Lomonosov, Combinatorial approaches to multiflow problems, *Appl. Discr. Math.* **11** (1985), 1–93.
13. L. Lovász and M. D. Plummer, “Matching Theory,” Akad. Kiado, Budapest, 1986.
14. W. Mader, Über die Maximalzahl kantendisjunkter A-Wege, *Arch. Math. (Basel)* **30** (1978), 325–336.